Applications of Fractional Calculus in Quantum Field Theory

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Abstract

In this work, I provide two results. The first result is the use of fractional calculus in providing a more rigorous foundation for dimensional regularization. I used the Riemann-Liouville fractional integral of order one-half to recover the volume of an n-ball and as special case of the Erdelyi-Kober operator to recover the surface area of an n-sphere. By writing the volume and surface area in terms of fractional calculus it provides a natural way of generalizing the volume and surface area of a sphere to non-integer dimension space, which I referred to as an α-sphere. This generalization recovers the formula used in dimensional regularization, which reinforces the current methods. The use of fractional derivatives, which is known to have a variety of equally valid definitions, also points to the possibility of formulating a variety of equally valid dimensional regularizations.

The second result is the use of the Baker-Haussdorf formula, a formula commonly used in quantum field theory, to establish a series formula for tempered fractional derivative, which is valid for all fractional derivatives.

Acknowledgements

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Contents

Chapter 1

Background

1.1 Fractional Calculus

Fractional calculus is the study of non-integer order integral and derivative operators, i.e. $\frac{\mathrm{d}^{0.5}}{\mathrm{d}x^{0.5}}$ [1]. In fact, the name "fractional" is a misnomer because we are also interested in non-rational order integral and derivative operators. A fractional derivative is a linear operator which satisfies

$$D_x^{\alpha}(D_x^{\beta} f(x)) = D_x^{\alpha+\beta} f(x), \tag{1.1}$$

where orders α and β are complex numbers and agree with the conventional derivative when the order is an integer [2]. Note that these conditions are *not* enough to identify a unique operator, so there are many different fractional derivatives and integrals. This is akin to the fact that the square root of a real number is not unique as there is ambiguity in the sign. A very popular fractional derivative is the Riemann-Liouville (RL) fractional integral, which is defined as

$$ {}^{\mathrm{RL}}_{\;a}D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\mathrm{d}t. \tag{1.2}$$

Figure (1.1) plots the function x, its first order integral $\frac{1}{2}x^2$, as well as a variety of intermediate orders.

The fractional Riemann-Liouville tempered derivative is a modification of the Riemann-Liouville fractional derivative (1.2) where the integrand is multiplied by an exponential factor [3, 4]

$$ {}^{\mathrm{RL}}_{\;a}D_x^{-\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x e^{-\lambda(t-x)}(t-x)^{\alpha-1} f(t)\mathrm{d}t. \tag{1.3}$$

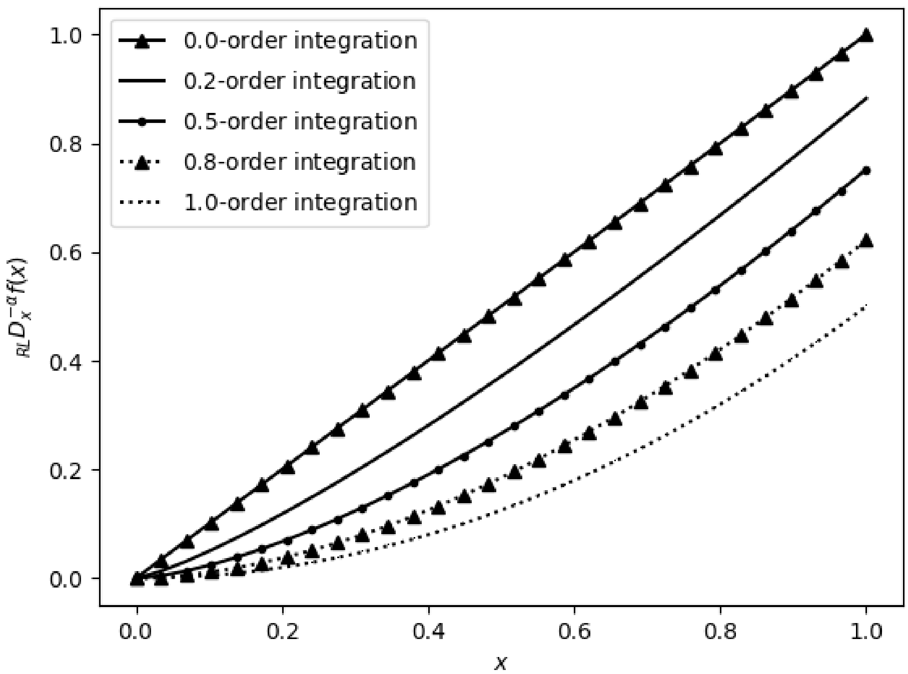

Figure 1.1: A series of Riemann-Liouville fractional integration of $f(x) = x$ with initial condition $a = 0$ using eq. (1.2). Notice that the 0^{th} order fractional integral does nothing to the function $f(x)$ and the 1^{st} order fractional integral agrees with conventional integral, namely $\int_0^x t\mathrm{d}t = \frac{1}{2}x^2$.

Throughout this thesis, I will use the following operator form,

$$ {}^{\mathrm{RL}}_{\;a}D_x^{-\alpha,\lambda} = e^{\lambda \hat{x}}\, {}^{\mathrm{RL}}_{\;a}D_x^{-\alpha} e^{-\lambda \hat{x}}. \tag{1.4} $$

1.2 Dimensional Regularization in Quantum Field Theory

Many integrals in quantum field theory diverge and can be resolved through regularization. One type of regularization is dimensional regularization, which is to take the dimension of the integral to be a real number in order to smoothly approach an integer such as $d = 4$.

Chapter 2

Dimensional Regularization in terms of Fractional Calculus

2.1 Introduction to Dimensional Regularization

The method of dimensional regularization aims to characterize the divergences in integrals found in quantum field theory using the notion of a non-integer order dimension [5–7]. As an example, let's consider the following integral

$$\int_{\mathbb{R}^d} \mathrm{d}^d q \frac{1}{(q^2+m^2)^2}. \tag{2.1}$$

More information can be found in [5] concerning the scattering process that this integral represents. Because the integrand is spherically symmetric, the integral can be rewritten in polar coordinates. The closed form formula for the surface area of an n-sphere is

$$S_n(r) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} r^n. \tag{2.2}$$

Then, our integral of interest becomes

$$\begin{aligned} &\int_0^\infty \mathrm{d}q S_{d-1}(q) \frac{1}{(q^2+m^2)^2} \\ = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} &\int_0^\infty \mathrm{d}q \frac{q^{d-1}}{(q^2+m^2)^2} \end{aligned} \tag{2.3}$$

Then, we consider the value of this integral when d no longer an integer which results in

$$\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})m^{d-4} \tag{2.4}$$

which is divergent when $d = 4$.

This marks the motivation for my work which is to address the sudden relaxation of d, which represents the number of independent variables in eq. (2.1), to a continuous value in eq. (2.4). Strictly speaking, eq. (2.2) is only valid for integer n, so it's not valid to simply relax d to a real or complex number. The goal of this chapter is to construct a stronger mathematical foundation for this method.

2.2 α-sphere, a generalization of n-spheres

This section provides two methods to arrive at one definition of the volume and surface area of a sphere in non-integer dimensions, which I dub as the α-sphere.

2.2.1 Method 1 of defining volume and surface area of spheres in non-integer space

The volume of an n-ball can be defined according to the following recurrence relation:

$$V_{n+1}(r) = \int_{-r}^{r} V_n(\sqrt{r^2 - x^2})\mathrm{d}x. \tag{2.5}$$

Coupled with the fact that $V_0(r) = 2r$, this recurrence relation can be used to generate all subsequent V_n. This equation can be interpreted geometrically as the volume of an $n+1$-ball is made up of slices of n-balls of varying radius. Let's consider the substitution $y = \sqrt{r^2 - x^2}$, which then can be manipulated into $x = \sqrt{r^2 - y^2}$:

$$V_{n+1}(r) = 2\int_0^r V_n(\sqrt{r^2 - x^2})\mathrm{d}x \tag{2.6}$$

$$\mathrm{d}x = \frac{-y}{r^2 - y^2}\mathrm{d}y \tag{2.7}$$

$$V_{n+1}(r) = 2\int_r^0 V_n(y)\frac{-y}{\sqrt{r^2 - y^2}}\mathrm{d}y \tag{2.8}$$

$$V_{n+1}(r) = 2\int_0^r V_n(y)\frac{y}{\sqrt{r^2 - y^2}}\mathrm{d}y \tag{2.9}$$

Note that $\mathrm{d}(y^2) = 2y\mathrm{d}y$, so

$$V_{n+1}(r) = \int_0^{y=r} V_n(y)\frac{1}{\sqrt{r^2 - y^2}}\,\mathrm{d}\big(y^2\big) \tag{2.10}$$

Equation (2.10) is of the same form as a Riemann Liouville fractional integral (1.2), so it can be rewritten as

$$V_{n+1}(r) = \Gamma(\frac{1}{2}) \cdot {}^{\mathrm{RL}}_{\;\;0}D^{-\frac{1}{2}}_{r^2} V_n(r). \tag{2.11}$$

The factor of $\Gamma(\frac{1}{2})$ is equal to $\sqrt{\pi}$ so,

$$\boxed{V_{n+1}(r) = \sqrt{\pi} \cdot {}^{\mathrm{RL}}_{\;\;0}D^{-\frac{1}{2}}_{r^2} V_n(r).} \tag{2.12}$$

With this, we have successfully redefined the recurrence relation for the volume of spheres (in integer dimensions) using fractional calculus. As an example, let's compute the volume of a sphere $V_3(r)$ using the area of a circle $V_2(r) = \pi r^2$.

$$\sqrt{\pi} \cdot {}^{\mathrm{RL}}_{\;\;0}D^{-\frac{1}{2}}_{r^2} V_2(r) \tag{2.13}$$

$$= \sqrt{\pi} \cdot {}^{\mathrm{RL}}_{\;\;0}D^{-\frac{1}{2}}_{r^2} (\pi r^2) \tag{2.14}$$

$$= \sqrt{\pi}\pi \cdot {}^{\mathrm{RL}}_{\;\;0}D^{-\frac{1}{2}}_{r^2} r^2 \tag{2.15}$$

Note that the Riemann-Liouville definition of fractional integration can be computed using a formula similar to the "power rule," i.e. ${}^{\mathrm{RL}}_{0}D_x^{-\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha}$.

$$=\sqrt{\pi}\pi\frac{\Gamma(1+1)}{\Gamma(\frac{3}{2}+1)}(r^2)^{\frac{3}{2}} \tag{2.16}$$

$$=\sqrt{\pi}\pi\frac{1}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}r^3 \tag{2.17}$$

$$=\frac{4\pi r^3}{3} = V_3(r). \tag{2.18}$$

Because eq. (2.12) is written in terms of fractional integration, it's natural to make use of the rule from equation (1.1) to extend the definition of n-spheres according to

$$\boxed{V_{n+\alpha}(r) = \pi^{\frac{\alpha}{2}} \cdot {}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{\alpha}{2}}V_n(r).} \tag{2.19}$$

I define the surface area by differentiating the volume with respect to r making use of the fact that a sphere is made up of many shells.

$$S_{\alpha+n-1}(r) = \frac{\mathrm{d}}{\mathrm{d}r}V_{\alpha+n}(r) \tag{2.20}$$

$$S_{\alpha+n-1}(r) = \pi^{\frac{\alpha}{2}}\frac{\mathrm{d}}{\mathrm{d}r}\,{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{\alpha}{2}}V_n(r) \tag{2.21}$$

Using the chain rule, $\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}r^2}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r^2} = 2r\frac{\mathrm{d}}{\mathrm{d}r^2}$.

$$S_{\alpha+n-1}(r) = \pi^{\frac{\alpha}{2}}2r\frac{\mathrm{d}}{\mathrm{d}r^2}\,{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{\alpha}{2}}V_n(r) \tag{2.22}$$

$$S_{\alpha+n-1}(r) = \pi^{\frac{\alpha}{2}}2r\cdot{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{\alpha}{2}}\frac{\mathrm{d}}{\mathrm{d}r^2}V_n(r) \tag{2.23}$$

$$S_{\alpha+n-1}(r) = \pi^{\frac{\alpha}{2}}2r\cdot{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{\alpha}{2}}\left(\frac{1}{2r}\frac{\mathrm{d}}{\mathrm{d}r}V_n(r)\right) \tag{2.24}$$

$$\boxed{S_{\alpha+n-1}(r) = \pi^{\frac{\alpha}{2}}r\cdot{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{\alpha}{2}}\left(r^{-1}S_{n-1}(r)\right)} \tag{2.25}$$

Note, that this is a special case of the Erdélyi-Kober operator [8].

2.2.2 Method 2 of defining spheres in non-integer space

Another way to achieve the same result is to begin with the following recurrence relation:

$$S_{n+2}(r) = 2\pi r \int_0^r S_n(r')\mathrm{d}r', \tag{2.26}$$

which requires the two initial conditions $S_0(r) = 2$ and $S_1(r) = 2\pi r$. In terms of operators, I refer to eq. (2.26) as

$$S_{n+2}(r) = 2\pi\hat{r} \cdot {}_0D_r^{-1} S_n(r). \tag{2.27}$$

where ${}_0D_r^{-1}$ is simply a shorthand for integration.

The goal of this section is to fractionalize the operator $2\pi\hat{r} \cdot {}_0D_r^{-1}$ in a way that is also consistent with $S_{n+1}(r) = \sqrt{2\pi\hat{r} \cdot {}_0D_r^{-1}} S_n(r)$. Finding roots of this operation by inspection is nontrivial, so I begin my analysis by funding the eigenfunctions of $\hat{r} \cdot {}_0D_r^{-1}$. Let $v(r)$ be a function with the power series expansion

$$v(r) = \sum_{k=0}^{\infty} a_k r^k. \tag{2.28}$$

Applying the operator of interest to $v(r)$ yields

$$\hat{r} \cdot {}_0D_r^{-1} v(r) = r \cdot \sum_{k=0}^{\infty} \frac{a_k}{k+1} r^{k+1} \tag{2.29}$$

$$\hat{r} \cdot {}_0D_r^{-1} v(r) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} r^{k+2}. \tag{2.30}$$

Letting $v(r)$ be an eigenfunction, we get

$$\hat{r} D_r^{-1} v(r) = \lambda v(r) \tag{2.31}$$

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} r^{k+2} = \lambda \sum_{k=0}^{\infty} a_k r^k, \tag{2.32}$$

Establishing that:

$$\frac{a_k}{k+1} = \lambda a_{k+2} \quad \text{for } k \geq 0 \tag{2.33}$$

This recurrence relation shows that there are two eigenfunctions:

$$v_0(r) = 1 + r^2 + \frac{r^4}{3} + \frac{r^6}{3 \cdot 5} + \ldots = \sum_{k=0}^{\infty} \frac{r^{2k}}{\lambda^k (2k-1)!!} \tag{2.34}$$

and

$$v_1(r) = r + \frac{r^3}{2} + \frac{r^5}{2 \cdot 4} + \ldots = \sum_{k=0}^{\infty} \frac{r^{2k+1}}{\lambda^k (2k)!!} \tag{2.35}$$

where !! denotes the double factorial [1]. The double factorial of an even number can be reduced to $(2k)!! = 2^k k!$, which simplifies the second eigenfunction to

$$v_1(r) = r \sum_{k=0}^{\infty} \frac{r^{2k}}{(2\lambda)^k k!} = r e^{\frac{1}{2\lambda} r^2} \tag{2.36}$$

The double factorial of an odd number can be reduced to $(2k+1)!! = \frac{2^k}{\sqrt{\pi}} \Gamma(k + \frac{1}{2})$, which simplifies the first eigenfunction to

$$v_0(r) = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{r^{2k}}{(2\lambda)^k \Gamma(k + \frac{1}{2})} = \sqrt{\pi} E_{1,\frac{1}{2}}(\frac{1}{2\lambda} r^2) \tag{2.37}$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function. The Mittag-Leffler function plays a very important role in fractional calculus [9]. This is very surprising as there is no fractional calculus to be seen as of yet. The Mittag-Leffler function is a generalization of the exponential function, with $E_{1,1}(x) = e^x$, and so the second eigenfunction can also be described using the Mittag-Leffler function as $v_1(r) = r E_{1,1}(\frac{1}{2\lambda} r^2)$. Note, that these are, strictly speaking, these are not eigenfunctions as there are extra terms relating to the constant of integration:

$$\hat{r} \cdot {}_0D_r^{-1} v_0(r) = \lambda v_0(r) - 1 \tag{2.38}$$

$$\hat{r} \cdot {}_0D_r^{-1} v_1(r) = \lambda v_1(r) - x \tag{2.39}$$

These two eigenfunctions are related according to [9]

$${}^{\mathrm{RL}}_{0}D_z^{-\alpha} (z^{\gamma-1} E_{\beta,\gamma})(a z^\beta) = z^{\alpha+\gamma-1} E_{\beta,\alpha+\gamma}(a z^\beta). \tag{2.40}$$

[1] The double factorial is distinct from evaluating the factorial twice. It is defined as $k!! = k(k-2)(k-4)\ldots$ and terminates at either 2 or 1, depending on the parity of k.

We can build a relationship between our two eigenfunctions if we allow $\alpha = \frac{1}{2}$, $\beta = 1$, $\gamma = \frac{1}{2}$, and $a = \frac{1}{2\lambda}$, then we get

$$
{}^{\mathrm{RL}}_{0}D_z^{-\frac{1}{2}}(\frac{1}{\sqrt{z}}E_{1,\frac{1}{2}}(\frac{1}{2\lambda}z)) = E_{1,1}(\frac{1}{2\lambda}z). \tag{2.41}
$$

Multiplying $\sqrt{z}$ on both sides gives:

$$
\sqrt{z}\cdot {}^{\mathrm{RL}}_{0}D_z^{-\frac{1}{2}}\frac{1}{\sqrt{z}}E_{1,\frac{1}{2}}(\frac{1}{2\lambda}z) = \sqrt{z}E_{1,1}(\frac{1}{2\lambda}z). \tag{2.42}
$$

Now, we can make the substitution $z = r^2$, which gives

$$
\hat{r}\cdot {}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}E_{1,\frac{1}{2}}(\frac{1}{2\lambda}r^2) = \hat{r}E_{1,1}(\frac{1}{2\lambda}r^2) \tag{2.43}
$$

demonstrating that the operator $\hat{r}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}$ brings the first eigenfunction to the second eigenfunction. For reasons which will be clear in a moment, let's divide both sides by $\frac{1}{\sqrt{2}}$ to get.

$$
\left(\frac{1}{\sqrt{2}}\hat{r}\cdot {}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}\right)E_{1,\frac{1}{2}}(\frac{1}{2\lambda}r^2) = \frac{1}{\sqrt{2}}\hat{r}E_{1,1}(\frac{1}{2\lambda}r^2) \tag{2.44}
$$

Performing this operator again gives.

$$
\left(\frac{1}{\sqrt{2}}\hat{r}\,{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}\right)\frac{1}{\sqrt{2}}\hat{r}E_{1,1}\left(\frac{1}{2\lambda}r^2\right) \tag{2.45}
$$

$$
=\frac{1}{2}\hat{r}\,{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{1}{2}}E_{1,1}\left(\frac{1}{2\lambda}r^2\right) \tag{2.46}
$$

$$
=\frac{1}{2}\hat{r}\sqrt{\hat{r}^2}E_{1,\frac{3}{2}}\left(\frac{1}{2\lambda}r^2\right) \tag{2.47}
$$

$$
=\frac{1}{2}\hat{r}^2E_{1,\frac{3}{2}}\left(\frac{1}{2\lambda}r^2\right) \tag{2.48}
$$

$$
=\lambda E_{1,\frac{1}{2}}\left(\frac{1}{2\lambda}r^2\right), \tag{2.49}
$$

yielding the first eigenfunction. In this case the particular eigenvalue is chosen to be $\lambda = 11$ to establish that $\frac{1}{\sqrt{2}}\hat{r}\,{}^{\mathrm{RL}}_{0}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}$ is a square root of the operator $\hat{r}\cdot {}_{0}D_r^{-1}$. Note that in fractional calculus, there is no chain rule, so

$D_{r^2}^{-\frac{1}{2}}$ cannot be easily simplified. This operator can be multiplied by $\sqrt{2\pi}$ to show that the operator

$$\sqrt{\pi}\hat{r}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1} \tag{2.50}$$

is a square root of $2\pi\hat{r}\cdot {}_0D_r^{-1}$.

2.2.3 A geometric interpretation of the α-sphere

Beginning with eq. (2.19) we can manipulate it into a form similar to eq. (2.5).

$$V_{n+\alpha}(r) = \pi^{\frac{\alpha}{2}} \cdot {}^{\mathrm{RL}}_{\ 0}D_{r^2}^{-\frac{\alpha}{2}} V_n(r) \tag{2.51}$$

$$V_{n+\alpha}(r) = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}\int_0^{y=r} V_n(y)(r^2-y^2)^{\frac{\alpha}{2}-1}\,\mathrm{d}\big(y^2\big) \tag{2.52}$$

$$V_{n+\alpha}(r) = -\frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}\int_0^{y=r} V_n(y)\mathrm{d}\big((r^2-y^2)^{\frac{\alpha}{2}}\big) \tag{2.53}$$

Let $x = (r^2-y^2)^{\frac{\alpha}{2}}$ which can be manipulated into $y = \sqrt{r^2 - x^{\frac{2}{\alpha}}}$.

$$V_{n+\alpha}(r) = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}\int_0^{x=r^\alpha} V_n\left(\sqrt{r^2-x^{\frac{2}{\alpha}}}\right)\mathrm{d}x \tag{2.54}$$

An ordinary n-ball is defined as the set of all points bounded by some particular radius, i.e. $\{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}|^2 \leq r^2\}$. The previous equation gives a geometric interpretation that the α-ball can be expressed similiarly, i.e. $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_{n-1}^2 + x_n^{\frac{2}{\alpha}} \leq r^2\}$ where $n = \lceil\alpha\rceil$. [2] A plot of a 1.5-ball is shown in figure (2.1). Note that the dimension that "receives the fractional dimension" grows slower than the other dimensions with respect to the radius i.e. the α ball will become more squished as the radius grows larger.

[2] $\lceil\cdot\rceil$ is the ceiling function.

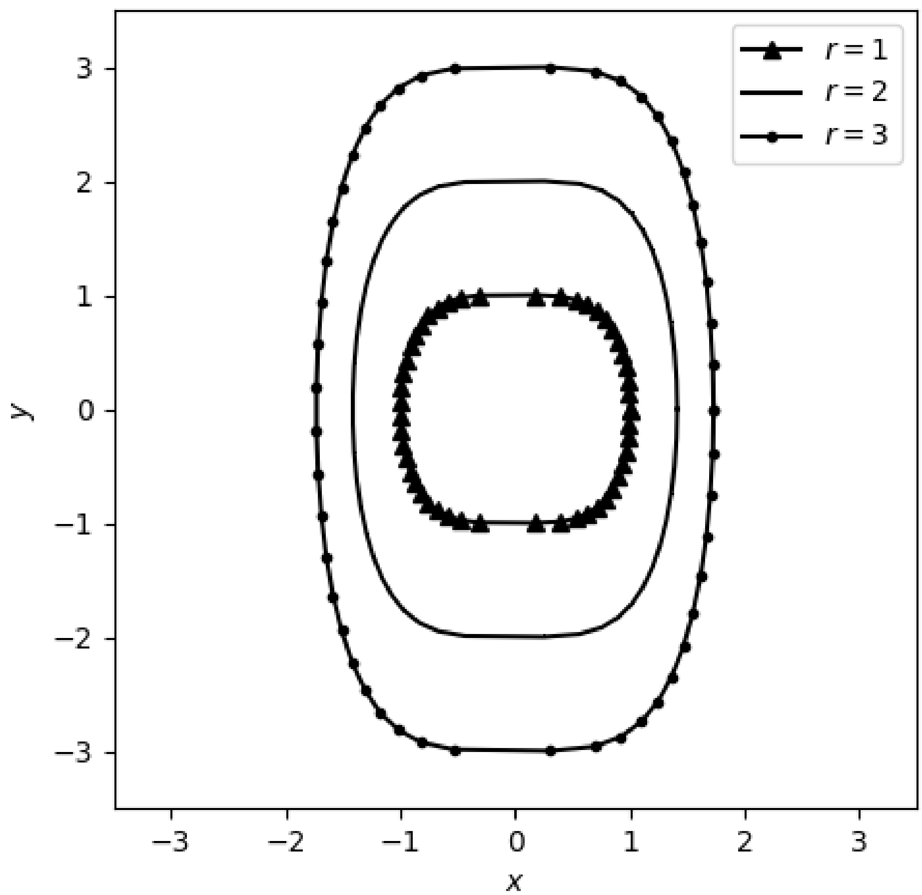

Figure 2.1: A plot of a 1.5-ball ranging from 1 to 3 as described in section 2.2.3 i.e. $x^{\frac{2}{\alpha}} + y^2 = r^2$.

2.3 Recovering the surface area of spheres in integer dimensions and the surface area factor used in dimensional regularization

We can use the $\sqrt{\pi}\hat{r}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}$ operator to recover $S_n(r)$. Let's begin with $S_1(r) = 2\pi r$.

$$\sqrt{\pi}\hat{r}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}S_1(r) \tag{2.55}$$

$$=\sqrt{\pi}\hat{r}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}2\pi r \tag{2.56}$$

$$=\sqrt{\pi}\hat{r}D_{r^2}^{-\frac{1}{2}}2\pi \tag{2.57}$$

$$=\sqrt{\pi}\hat{r}2\pi\frac{\sqrt{r^2}\Gamma(1)}{\Gamma(\frac{1}{2}+1)} \tag{2.58}$$

$$=\sqrt{\pi}\hat{r}2\pi\frac{2r}{\sqrt{\pi}} \tag{2.59}$$

$$=\hat{r}2\pi\cdot 2r \tag{2.60}$$

$$=4\pi r^2 = S_2(r) \tag{2.61}$$

This accurately recovers the surface area of a sphere.

Note that there is a case of which needs special attention, namely incrementing from $S_0(r) = 2$ to $S_1(r) = 2\pi r$. This is because this particular case needs to evaluate $I_x^{\frac{1}{2}}x^{-\frac{1}{2}}$, which cannot be computed by the power rule. This is akin to integrating $\frac{1}{x}$.

Using fractional calculus, we can generalize $\sqrt{\pi}\hat{r}D_{r^2}^{-\frac{1}{2}}\hat{r}^{-1}$ to $\pi^{\frac{\alpha}{2}}\hat{r}D_{r^2}^{-\frac{\alpha}{2}}\hat{r}^{-1}$. If we begin with $S_1 = 2\pi r$, then we get that

$$S_{\alpha+1}(r) = \pi^{\frac{\alpha}{2}}\hat{r}D_{r^2}^{-\frac{\alpha}{2}}\hat{r}^{-1}\cdot 2\pi r \tag{2.62}$$

$$S_{\alpha+1}(r) = \pi^{\frac{\alpha}{2}}\hat{r}D_{r^2}^{-\frac{\alpha}{2}}2\pi \tag{2.63}$$

$$S_{\alpha+1}(r) = 2\pi^{\frac{\alpha}{2}+1}\hat{r}(r^2)^{\frac{\alpha}{2}}\frac{\Gamma(1)}{\Gamma(\frac{\alpha}{2}+1)} \tag{2.64}$$

$$S_{\alpha+1}(r) = \frac{2\pi^{\frac{\alpha}{2}+1}}{\Gamma(\frac{\alpha}{2}+1)}r^{\alpha+1} \tag{2.65}$$

which recovers eq. (2.2).

Chapter 3

Baker-Hausdorff Theorem and Fractional Tempered Derivatives

The tempered Riemann-Liouville fractional derivative, defined in eq. (1.4), is of the forl of the left-hand-side of the Baker-Hausdorff Theorem: [10]

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]\dots \tag{3.1}$$

where $\hat{A}$ and $\hat{B}$ are operators and the commutation is defined as $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Substituting $\hat{A}$ for $-\lambda\hat{x}$ and $\hat{B}$ for D_x^α, the first commutation yields

$$[-\lambda\hat{x}, D_x^\alpha]f = -\lambda\hat{x}D_x^\alpha f - D_x^\alpha(-\lambda x f), \tag{3.2}$$

where f is a "dummy function" for clarity. Note D_x^α, for this section, is considered to be any arbitrary fractional derivative because the following proof does not specifically require the Riemann-Liouville definition. The right hand side can be expanded using the Leibniz product rule for fractional derivatives, which is valid for all fractional derivatives [1].

$$D_x^\alpha(-\lambda x f) = -\lambda D_x^\alpha(xf) = -\lambda\sum_{j=0}^{\infty}\binom{\alpha}{j}(D_x^{\alpha-j}f)(D_x^j(-\lambda x)) \tag{3.3}$$

where $\binom{a}{b}$ is the binomial coefficient. Because the factor of $D_x^j x$ is nilpotent, this series only has two nonzero terms when $j=0$ and $j=1$.

$$D_x^\alpha(-\lambda x f) = -\lambda\left(\binom{\alpha}{0}(D_x^{\alpha-0}f)(D_x^0 x) + \binom{\alpha}{1}(D_x^{\alpha-1}f)(D_x^1 x)\right). \tag{3.4}$$

Simplifying yields

$$D_x^\alpha(-\lambda x f) = -\lambda(x D_x^\alpha f + \alpha D_x^{\alpha-1} f). \tag{3.5}$$

Substituting back into eq. (3.2) yields

$$[-\lambda\hat{x}, D_x^\alpha]f = -\lambda\hat{x}D_x^\alpha f + \lambda(x D_x^\alpha f + \alpha D_x^{\alpha-1} f), \tag{3.6}$$

$$[-\lambda\hat{x}, D_x^\alpha]f = \lambda\alpha D_x^{\alpha-1} f, \tag{3.7}$$

and rewriting as an operator, without f:

$$[-\lambda\hat{x}, D_x^\alpha] = \lambda\alpha D_x^{\alpha-1}. \tag{3.8}$$

It can be shown through induction that

$$[-\lambda\hat{x}, D_x^\alpha]_k = \lambda^k(\alpha)_k D_x^{\alpha-k}, \tag{3.9}$$

where $(\alpha)_k$ is the falling factorial, and $[\cdot]_k$ refers to a k^{th} order commutation. A detailed proof can be found in the appendix in section A. Substituting into eq. (3.1), we get

$$e^{-\lambda x} D_x^\alpha e^{\lambda x} = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k(\alpha)_k D_x^{\alpha-k}. \tag{3.10}$$

The factor $\frac{1}{k!}(\alpha)_k$ can be substituted for $\binom{\alpha}{k}$

$$e^{-\lambda x} D_x^\alpha e^{\lambda x} = \sum_{k=0}^{\infty} \binom{\alpha}{k}\lambda^k D_x^{\alpha-k}. \tag{3.11}$$

This series is very similiar to Newton's generalized binomial theorem, which states

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k \tag{3.12}$$

where x, y, and α are real and $|x| > |y|$ for convergence. The difference being that eq. (3.11) includes operators. This is to say that, in some sense, $e^{-\lambda x} D_x^\alpha e^{\lambda x}$ has a strong relationship to the expression $(\lambda + \frac{\mathrm{d}}{\mathrm{d}x})^\alpha$. This relationship is can be reinforced by Herrmann's claim [1], that following relationship holds for small λ,

$$e^{-\lambda x}\,{}_{\mathrm{RL}}D_x^\alpha e^{\lambda x} f(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \lambda^{\alpha-k} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \tag{3.13}$$

which is similar to the right hand side of equation (3.11), but the order of the derivative and the power of λ is switched.

Chapter 4

Conclusions

In Chapter 2, I give a strong foundation for the methods of dimensional regularization by reproducing the existing formulas for the surface area of a sphere of a non-integer dimension using fractional derivatives. The fractional operator was also rewritten to give a geometric picture of what a non-integer dimension sphere may be understood as.

There are two particularly interesting things to note from this work. First, the spheres of integer dimensions are still described using fractional derivatives i.e. Riemann Liouville half-integration. This indicates that the usage of fractional calculus is not simply an addendum to n-sphere, but is deeply related to spheres. Second, it's well known that there are a variety of equally valid definitions for a fractional derivative and the Riemann-Liouville definition is merely just one of them. This work does not rule out the possibility that another fractional derivative may generalize the n-sphere, which would be a natural continuation of this work. Furthermore, one might ask whether or not other such could produce an alternative dimensional regularization.

In Chapter 3, I used the Baker-Hausdorff theorem to produce a series representation for all tempered fractional derivatives. Because of it's stark similarity to the binomial formula, future work may be able to make the expression $(\lambda + \frac{\mathrm{d}}{\mathrm{d}x})^\alpha$ rigorous.

These two results show that fractional calculus has an important place in quantum field theory and invites collaboration between these two communities.

Appendix A

Derivation of eq. (3.9)

The left-hand-side of eq. (3.9) is defined as

$$[-\lambda\hat{x}, D_x^\alpha]_k = [-\lambda\hat{x}, [-\lambda\hat{x}, D_x^\alpha]_{k-1}] \tag{A.1}$$

where k is a natural number and $[-\lambda\hat{x}, D_x^\alpha]_1 = [-\lambda\hat{x}, D_x^\alpha]$.

Proof. Equation (3.8) serves as the base case for this inductive proof. To prove eq. (3.9), let's assume the claim as an inductive hypothesis.

$$[-\lambda\hat{x}, D_x^\alpha]_{k+1} = [-\lambda\hat{x}, [-\lambda\hat{x}, D_x^\alpha]_k] \tag{A.2}$$

$$[-\lambda\hat{x}, D_x^\alpha]_{k+1} = [-\lambda\hat{x}, \lambda^k(\alpha)_k D_x^{\alpha-k}] \tag{A.3}$$

Because $\hat{x}$ and D_x^α are linear operators, the constant $\lambda^k(\alpha)_k$ can be pulled out of the commutation:

$$[-\lambda\hat{x}, D_x^\alpha]_{k+1} = \lambda^k(\alpha)_k[-\lambda\hat{x}, D_x^{\alpha-k}] \tag{A.4}$$

Using, eq. (3.8):

$$[-\lambda\hat{x}, D_x^\alpha]_{k+1} = \lambda^k(\alpha)_k \cdot \lambda(\alpha - k)D_x^{\alpha-k-1} \tag{A.5}$$

$$[-\lambda\hat{x}, D_x^\alpha]_{k+1} = \lambda^{k+1}(\alpha)_{k+1}D_x^{\alpha-(k+1)} \tag{A.6}$$

□

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